高维概率

High-Dimensional Probability

十、随机矩阵Concentration

@滕佳烨

- **上节课说了啥**

1. 随机矩阵的奇异值、模(2-norm F-norm)
2. 随机矩阵的各项同性

- **这节课要说啥**

随机矩阵在**各种情形**下的Concentration

各种情形？

Coordinate-wise sub-Gaussian

Coordinate-wise but not sub-Gaussian

Row-wise sub-Gaussian

Symmetric but not sub-Gaussian

Concentration?

$$\|A\| = s_1(A)$$

- **这节课要说啥**

随机矩阵在**各种情形**下的Concentration

各种情形？

- Coordinate-wise sub-Gaussian
- Coordinate-wise but not sub-Gaussian
- Row-wise sub-Gaussian
- Symmetric but not sub-Gaussian

Concentration?

$$\|A\| = s_1(A)$$

1. Coordinate-wise sub-Gaussian (单边)

Theorem 4.4.5 (Norm of matrices with sub-gaussian entries). *Let A be an $m \times n$ random matrix whose entries A_{ij} are independent, mean zero, sub-gaussian random variables. Then, for any $t > 0$ we have*[6]

$$\|A\| \le CK\left(\sqrt{m} + \sqrt{n} + t\right)$$

with probability at least $1 - 2\exp(-t^2)$. Here $K = \max_{i,j} \|A_{ij}\|_{\psi_2}$.

- 从就元素而言，元素必须是次高斯

1. Coordinate-wise sub-Gaussian (单边)

Exercise 4.4.6 (Expected norm). ☕ Deduce from Theorem 4.4.5 that

$$\mathbb{E}\,\|A\| \le CK\left(\sqrt{m}+\sqrt{n}\right).$$

Exercise 4.4.7 (Optimality). ☕☕ Suppose that in Theorem 4.4.5 the entries A_{ij} have unit variances. Prove that

$$\mathbb{E}\,\|A\| \ge C\left(\sqrt{m}+\sqrt{n}\right).$$

- 期望形式，是紧的

2. Coordinate-wise（非次高斯）

Exercise 6.5.2 (Rectangular matrices). Let A be an $m \times n$ random matrix whose entries are independent, mean zero random variables. Show that

$$\mathbb{E}\,\|A\| \le C\sqrt{\log(m+n)}\,\Big(\mathbb{E}\max_i \|A_i\|_2 + \mathbb{E}\max_j \|A^j\|_2\Big)$$

where A_i and A^j denote the rows and columns of A, respectively.

Exercise 6.5.3 (Sharpness). Show that the result of Exercise 6.5.2 is sharp up to the logarithmic factor, i.e. one always has

$$\mathbb{E}\,\|A\| \ge c\Big(\mathbb{E}\max_i \|A_i\|_2 + \mathbb{E}\max_j \|A^j\|_2\Big).$$

- 虽然不紧，但是只差了一个log项

3. Row-wise sub-Gaussian (双边)

Theorem 4.6.1 (Two-sided bound on sub-gaussian matrices). *Let A be an $m \times n$ matrix whose rows A_i are independent, mean zero, sub-gaussian isotropic random vectors in $\mathbb{R}^n$. Then for any $t \geq 0$ we have*

$$\sqrt{m} - CK^2(\sqrt{n} + t) \leq s_n(A) \leq s_1(A) \leq \sqrt{m} + CK^2(\sqrt{n} + t) \tag{4.21}$$

with probability at least $1 - 2\exp(-t^2)$. Here $K = \max_i \|A_i\|_{\psi_2}$.

- 有对奇异值上界、下界的估计（双边）
- 只需要row之间独立，不需要每个单元都独立。注意到这更符合一般数据的要求。
- 当m>>n，则该矩阵是一个近似isometry.

3. Row-wise sub-Gaussian (双边)

Exercise 4.6.3. ☕☕ Deduce from Theorem 4.6.1 the following bounds on the expectation:

$$\sqrt{m} - CK^2\sqrt{n} \le \mathbb{E}\, s_n(A) \le \mathbb{E}\, s_1(A) \le \sqrt{m} + CK^2\sqrt{n}.$$

- 对应的均值形式。

4. Symmetric-wise

Theorem 6.5.1 (Norms of random matrices with non-i.i.d. entries). *Let A be an $n \times n$ symmetric random matrix whose entries on and above the diagonal are independent, mean zero random variables. Then*

$$\mathbb{E}\,\|A\| \le C\sqrt{\log n} \cdot \mathbb{E}\max_i \|A_i\|_2,$$

where A_i denote the rows of A.

$$\mathbb{E}\,\|A\| \ge \mathbb{E}\max_i \|A_i\|_2.$$

- 虽然不紧，但是只差了一个log项

小总结

*-wise	sub-Gaussian	双边?	只有期望形式	紧的
Coordinate	√	×	×	√
Coordinate	×	×	√	×
Row	√	√	×	√
Symmetry	×	×	√	×

第一种与第三种的证明

$$\|A\| = \max_{x \in S^{n-1}} \|Ax\|_2$$

一共多少碎片？

Covering number

- S:一个*连续*的球壳

↓

离散化：把球切成一个个小碎片

↓

碎片比较小，每个碎片的性质可以由一个点来进行估计

↓

Union Bound，把碎片粘回一起

1. **四种情形下的Concentration**

   Coordinate, row, symmetry

   sub-Gaussian?

   期望形式?

   双边?

2. **证明思路：离散化**

   Covering number

谢谢！

@滕佳烨