Concentration without dependency
——function concentration

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前情回顾:

Concentration of random variable

- Hoeffding Inequality, Bernstein Inequality

Concentration of random vector

- Norm concentration

Concentration of random matrix

- Eigenvalue concentration

They are all independent regimes!

Independent entries; Independent rows…

思考：

$X \sim N(0, I_n)$, $f: \mathbb{R}^n \to \mathbb{R}$, w.h.p

$$f(X) \approx \mathbb{E} f(X)$$

1. If $f = a^T x$ is linear, easy since $f(X)$ is gaussian distribution.
2. If f is non-linear…Not always true, especially when f oscillate wildly

   当 f 的波动非常大，那么它可能离均值就会很远

   例如，X本身sub-Gaussian, 而f(X) 非sub-Gaussian

   怎么办？对 f 的波动做出限制！

Lipschitz Functions (w.r.t. some distance metric!)

Definition 5.1.1 (Lipschitz functions)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called *Lipschitz* if there exists $L \in \mathbb{R}$ such that

$$d_Y(f(u), f(v)) \leq L \cdot d_X(u, v) \quad \text{for every } u, v \in X.$$

The infimum of all L in this definition is called the *Lipschitz norm* of f and is denoted $\|f\|_{\mathrm{Lip}}$.

核心：Lipschitz 函数跑的不能比线性快

(a) Every Lipschitz function is uniformly continuous.

(b) Every differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, and

$$\|f\|_{\mathrm{Lip}} \leq \sup_{x \in \mathbb{R}^n} \|\nabla f(x)\|_2.$$

导数有界 （强条件）

→Lipschitz连续

→ 一致连续

Concentration on the sphere with Euclidean metric

Theorem 5.1.4 (Concentration of Lipschitz functions on the sphere). *Consider a random vector* $\boxed{X \sim \mathrm{Unif}(\sqrt{n}S^{n-1}),}$ *i.e.* X *is uniformly distributed on the Euclidean sphere of radius* $\sqrt{n}$. *Consider a* $\boxed{\textit{Lipschitz function}}$[1] $f : \sqrt{n}S^{n-1} \to \mathbb{R}$. *Then*

$$\|f(X) - \mathbb{E} f(X)\|_{\psi_2} \le C\|f\|_{\mathrm{Lip}}.$$

Using the definition of the sub-gaussian norm, the conclusion of Theorem 5.1.4 can be stated as follows: for every $t \ge 0$, we have

$$\mathbb{P}\left\{|f(X) - \mathbb{E} f(X)| \ge t\right\} \le 2\exp\Big(-\frac{ct^2}{\|f\|_{\mathrm{Lip}}^2}\Big).$$

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Hint:

1. 证明对均值的concentration，等价于证明对中位数的concentration
2. 寻找到$f(X) \leq M$的X集合, 如果我们将其扩充到 $f(X') = f(X) + [f(X') - f(X)] \leq M + t$ [Here we use the fact that $\|f\|_{Lip} = 1$ and $\|X' - X\| \leq t$]，将会产生一个指数的界（0-1律）

Concentration on the Gaussian RV with Euclidean metric

Theorem 5.2.2 (Gaussian concentration). *Consider a random vector* $X \sim N(0, I_n)$ *and a Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *(with respect to the Euclidean metric). Then*

$$\|f(X) - \mathbb{E}\, f(X)\|_{\psi_2} \leq C\|f\|_{\mathrm{Lip}}. \tag{5.7}$$

Concentration on the Bernoulli RV with Hamming metric

Theorem 5.2.5 (Concentration on the Hamming cube). *Consider a random vector* $X \sim \mathrm{Unif}\{0,1\}^n$. *(Thus, the coordinates of* X *are independent* $\mathrm{Ber}(1/2)$ *random variables.) Consider a function* $f : \{0,1\}^n \to \mathbb{R}$. *Then*

$$\|f(X) - \mathbb{E}\, f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\mathrm{Lip}}}{\sqrt{n}}. \tag{5.8}$$

Take-away Messages

1. $f(X) \to \mathbb{E}f(X)$ with f Lipschitz and $X \sim \mathbb{P}$
2. Lipschitz: which metric?
3. $X \sim \mathbb{P}$, which distribution?
   - sphere with Euclidean metric
   - Gaussian RV with Euclidean metric
   - Bernoulli RV with Hamming metric

Thanks!

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Thanks!