HDP
(17) Something on Random Process

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Random Process $(X_t)_{t\in T}$

e.g., Gaussian Process $$X_t = \langle g, t\rangle, \quad t \in T,$$

特点：*随机变量函数*

随机变量(g)：函数的每一点都具有随机性

函数性(t)：函数的自变量，两个时间中可能会有correlation

性质：

$$\mathbb{E}\, X_t = 0 \quad \text{for all } t \in T.$$

$$\Sigma(t,s) := \operatorname{cov}(X_t, X_s) = \mathbb{E}\, X_t X_s,$$

$$d(t,s) := \|X_t - X_s\|_{L^2} = \left(\mathbb{E}(X_t - X_s)^2\right)^{1/2}, \quad t,s \in T.$$

Gaussian Process $(\boldsymbol{X_t})_{\boldsymbol{t\in T}}$ （类比高斯随机变量）

For every time subset T_0, $(X_t)_{t\in T_0}$ is Gaussian.

Canonical Gaussian Process

$$X_t := \langle g, t \rangle, \quad t \in T.$$

One can transform any Gaussian Process to a canonical Gaussian Process.

$$d(s,t) = \|s - t\|_2$$

Lemma 7.1.12 (Gaussian random vectors)**.** *Let Y be a mean zero Gaussian random vector in $\mathbb{R}^n$. Then there exist points $t_1, \ldots, t_n \in \mathbb{R}^n$ such that*

$$Y \equiv (\langle g, t_i \rangle)_{i=1}^n, \quad \text{where } g \sim N(0, I_n).$$

Here "$\equiv$" means that the distributions of the two random vectors are the same.

The interesting metric in random process

$$\mathbb{E} \sup_{t \in T} X_t$$

Lem: reflection principle.

For standard Brownian motion (special GP),

$$\mathbb{E} \sup_{t \le t_0} X_t = \sqrt{\frac{2t_0}{\pi}} \quad \text{for every } t_0 \ge 0.$$

For general GP…

Theorem 7.2.1 (Slepian's inequality). *Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two mean zero Gaussian processes. Assume that for all $t, s \in T$, we have*

$$\boxed{\mathbb{E}\, X_t^2 = \mathbb{E}\, Y_t^2 \quad \textit{and} \quad \mathbb{E}(X_t - X_s)^2 \le \mathbb{E}(Y_t - Y_s)^2.} \tag{7.2}$$

Then for every $\tau \in \mathbb{R}$ we have

$$\mathbb{P}\left\{\sup_{t\in T} X_t \ge \tau\right\} \le \mathbb{P}\left\{\sup_{t\in T} Y_t \ge \tau\right\}. \quad \mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t.$$

Intuition. 随机过程的波动由二阶信息决定。
平方项二阶信息相等，交叉项 X 的correlation更大。
X_t 比 Y_t 的波动要小，因此最大值更小。

Hint: Gaussian interpolation

Technique: Gaussian interpolation (二维降一维)

$$Z(u) := \sqrt{u}\, X + \sqrt{1-u}\, Y, \quad u \in [0,1].$$

性质:

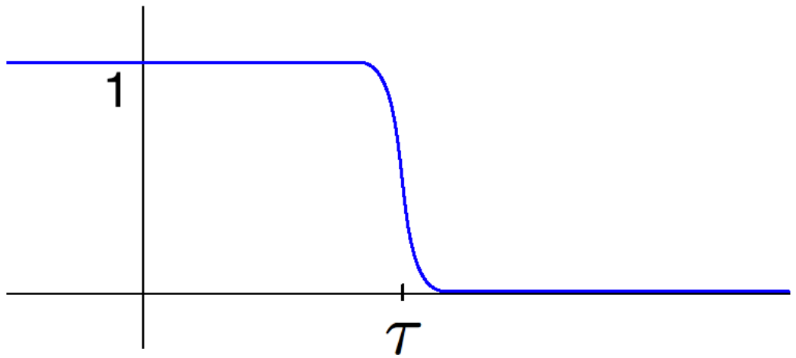
$$\frac{d}{du}\,\mathbb{E}\, f(Z(u)) = \frac{1}{2}\sum_{i,j=1}^{n} (\Sigma_{ij}^X - \Sigma_{ij}^Y)\,\mathbb{E}\left[\frac{\partial^2 f}{\partial x_i\, \partial x_j}(Z(u))\right].$$

[Use the property of Gaussian variable] $\mathbb{E}\, f'(X) = \mathbb{E}\, X f(X).$

Choose a proper f

$$f(x) = h(x_1)\cdots h(x_n).$$

$$f(x) \approx \mathbf{1}_{\{\max_i x_i < \tau\}}.$$



Take-away Messages

1. Random Process (expectation, covariance, increment)
2. Gaussian Process
    - Any Gaussian Process can be transferred to Canonical GP
    - $d(s,t) = \|s-t\|_2$
3. Consider $\mathbb{E}\sup_{t\in T} X_t$ for GP: Slepian's inequality
    - Comparison between X_t and Y_t using second order information
    - Small fluctuation leads to small expectation.

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Thanks!