HDP
(18) Analysis on Gaussian Process

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前情回顾

1. Random Process (expectation, covariance, increment)
2. Gaussian Process
    - Any Gaussian Process can be transferred to Canonical GP
    - $d(s,t) = \|s-t\|_2$
3. Consider $\mathbb{E}\sup_{t\in T} X_t$ for GP: Slepian's inequality
    - Comparison between X_t and Y_t using second order information
    - Small fluctuation leads to small expectation.

- Today, we talk about other inequalities about $\mathbb{E}\sup_{t\in T} X_t$

Theorem 7.2.1 (Slepian's inequality). *Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two mean zero Gaussian processes. Assume that for all $t, s \in T$, we have*

$$\mathbb{E} X_t^2 = \mathbb{E} Y_t^2 \quad \text{and} \quad \mathbb{E}(X_t - X_s)^2 \leq \mathbb{E}(Y_t - Y_s)^2. \tag{7.2}$$

Then for every $\tau \in \mathbb{R}$ we have

$$\mathbb{P}\left\{\sup_{t\in T} X_t \geq \tau\right\} \leq \mathbb{P}\left\{\sup_{t\in T} Y_t \geq \tau\right\}. \quad \mathbb{E}\sup_{t\in T} X_t \leq \mathbb{E}\sup_{t\in T} Y_t.$$

Intuition. X_t 比 Y_t 的波动要小，因此最大值更小。

是否能用更弱的条件推出类似的结论？ Sudakov-Fernique's inequality

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Theorem 7.2.11 (Sudakov-Fernique's inequality). *Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two mean zero Gaussian processes. Assume that for all $t, s \in T$, we have*

$$\mathbb{E}(X_t - X_s)^2 \le \mathbb{E}(Y_t - Y_s)^2.$$

Then

$$\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t.$$

Hint: Interpolation.
choose f as

$$f(x) := \frac{1}{\beta}\log\sum_{i=1}^n e^{\beta x_i}.$$

Two-dim extension: Gordon's inequality

Exercise 7.2.14 (Gordon's inequality). ☕☕☕ Prove the following extension of Slepian's inequality due to Y. Gordon. Let $(X_{ut})_{u\in U,\, t\in T}$ and $Y = (Y_{ut})_{u\in U,\, t\in T}$ be two mean zero Gaussian processes indexed by pairs of points (u, t) in a product set $U \times T$. Assume that we have

$$\mathbb{E}\, X_{ut}^2 = \mathbb{E}\, Y_{ut}^2, \quad \mathbb{E}(X_{ut} - X_{us})^2 \le \mathbb{E}(Y_{ut} - Y_{us})^2 \quad \text{for all } u, t, s;$$
$$\mathbb{E}(X_{ut} - X_{vs})^2 \ge \mathbb{E}(Y_{ut} - Y_{vs})^2 \quad \text{for all } u \neq v \text{ and all } t, s.$$

Then for every $\tau \ge 0$ we have

$$\mathbb{P}\left\{\inf_{u\in U} \sup_{t\in T} X_{ut} \ge \tau\right\} \le \mathbb{P}\left\{\inf_{u\in U} \sup_{t\in T} Y_{ut} \ge \tau\right\}.$$

Consequently,

$$\mathbb{E} \inf_{u\in U} \sup_{t\in T} X_{ut} \le \mathbb{E} \inf_{u\in U} \sup_{t\in T} Y_{ut}. \tag{7.12}$$

Another geometric view: Sudakov's minoration inequality

Theorem 7.4.1 (Sudakov's minoration inequality). *Let* $(X_t)_{t\in T}$ *be a mean zero Gaussian process. Then, for any* $\varepsilon \geq 0$, *we have*

$$\mathbb{E} \sup_{t\in T} X_t \geq c\varepsilon \sqrt{\log \mathcal{N}(T, d, \varepsilon)}.$$

where d *is the canonical metric defined in* (7.13).

Intuition: increment $d(t, s)$ contains all the information of GP

Hint: Consider a set T with only N different points. Since every two points are far from each other, they have a large variance (related to ϵ). Comparing it with canonical GP reaches the conclusion.

Sudakov's inequality (two-side)

Theorem 8.1.13 (Two-sided Sudakov's inequality). *Let $T \subset \mathbb{R}^n$ and set*

$$s(T) := \sup_{\varepsilon \geq 0} \varepsilon \sqrt{\log \mathcal{N}(T, \varepsilon)}.$$

Then

$$c \cdot s(T) \leq w(T) \leq C \log(n) \cdot s(T).$$

Take-away Messages

Analyze $\mathbb{E}\sup_{t\in T} X_t$ for Gaussian Process

1. Slepian's inequality: Small fluctuation leads to small expectation.
2. Sudakov-Fernique's inequality: remove variance requirement (only expectation results)
3. Gordon's inequality: two-dim extension

4. *Sudakov's minoration inequality: lower bound, Geometric

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Thanks!