HDP (18) Analysis on Gaussian Process



前情回顾

- 1. Random Process (expectation, covariance, increment)
- 2. Gaussian Process
 - Any Gaussian Process can be transferred to Canonical GP
 - $d(s,t) = ||s-t||_2$
- 3. Consider $\mathbb{E} \sup_{t \in T} X_t$ for GP: Slepian's inequality
 - Comparison between X_t and Y_t using second order information
 - Small fluctuation leads to small expectation.
- Today, we talk about other inequalities about $\mathbb{E} \sup_{t \in T} X_t$

Theorem 7.2.1 (Slepian's inequality). Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two mean zero Gaussian processes. Assume that for all $t, s \in T$, we have

$$\mathbb{E} X_t^2 = \mathbb{E} Y_t^2 \quad and \quad \mathbb{E} (X_t - X_s)^2 \le \mathbb{E} (Y_t - Y_s)^2.$$
(7.2)

Then for every $\tau \in \mathbb{R}$ we have

$$\mathbb{P}\left\{\sup_{t\in T} X_t \ge \tau\right\} \le \mathbb{P}\left\{\sup_{t\in T} Y_t \ge \tau\right\}. \quad \mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t.$$

Intuition. X_t 比 Y_t 的波动要小,因此最大值更小。

是否能用更弱的条件推出类似的结论? Sudakov-Fernique's inequality

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Theorem 7.2.11 (Sudakov-Fernique's inequality). Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two mean zero Gaussian processes. Assume that for all $t, s \in T$, we have

$$\mathbb{E}(X_t - X_s)^2 \le \mathbb{E}(Y_t - Y_s)^2.$$

 $\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t.$

Hint: Interpolation. choose f as

$$f(x) := \frac{1}{\beta} \log \sum_{i=1}^{n} e^{\beta x_i}.$$

Then

Two-dim extension: Gordon's inequality

Exercise 7.2.14 (Gordon's inequality). $\blacksquare \blacksquare \blacksquare$ Prove the following extension of Slepian's inequality due to Y. Gordon. Let $(X_{ut})_{u \in U, t \in T}$ and $Y = (Y_{ut})_{u \in U, t \in T}$ be two mean zero Gaussian processes indexed by pairs of points (u, t) in a product set $U \times T$. Assume that we have

$$\mathbb{E} X_{ut}^2 = \mathbb{E} Y_{ut}^2, \quad \mathbb{E} (X_{ut} - X_{us})^2 \leq \mathbb{E} (Y_{ut} - Y_{us})^2 \quad \text{for all } u, t, s;$$
$$\mathbb{E} (X_{ut} - X_{vs})^2 \geq \mathbb{E} (Y_{ut} - Y_{vs})^2 \quad \text{for all } u \neq v \text{ and all } t, s.$$

Then for every $\tau \geq 0$ we have

$$\mathbb{P}\left\{\inf_{u\in U}\sup_{t\in T}X_{ut}\geq \tau\right\}\leq \mathbb{P}\left\{\inf_{u\in U}\sup_{t\in T}Y_{ut}\geq \tau\right\}.$$

Consequently,

$$\mathbb{E}\inf_{u\in U}\sup_{t\in T}X_{ut} \leq \mathbb{E}\inf_{u\in U}\sup_{t\in T}Y_{ut}.$$

Another geometric view: Sudakov's minoration inequality

Theorem 7.4.1 (Sudakov's minoration inequality). Let $(X_t)_{t\in T}$ be a mean zero Gaussian process. Then, for any $\varepsilon \geq 0$, we have

$$\mathbb{E}\sup_{t\in T} X_t \ge c\varepsilon \sqrt{\log \mathcal{N}(T, d, \varepsilon)}.$$

where d is the canonical metric defined in (7.13).

Intuition: increment d(t, s) contains all the information of GP

Hint: Consider a set *T* with only N different points. Since every two points are far from each other, they have a large variance (related to ϵ). Comparing it with canonical GP reaches the conclusion.

Sudakov's inequality (two-side)

Theorem 8.1.13 (Two-sided Sudakov's inequality). Let $T \subset \mathbb{R}^n$ and set

$$s(T) := \sup_{\varepsilon \ge 0} \varepsilon \sqrt{\log \mathcal{N}(T, \varepsilon)}.$$

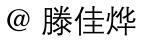
Then

$$c \cdot s(T) \le w(T) \le C \log(n) \cdot s(T).$$

Take-away Messages

Analyze $\mathbb{E} \sup_{t \in T} X_t$ for Gaussian Process

- 1. Slepian's inequality: Small fluctuation leads to small expectation.
- 2. Sudakov-Fernique's inequality: remove variance requirement (only expectation results)
- 3. Gordon's inequality: two-dim extension
- 4. *Sudakov's minoration inequality: lower bound, Geometric



Thanks!