HDP

(21) Generic Chaining Bound

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前情回顾

1. Dudley's inequality. The upper bound of $\mathbb{E} \sup_{t \in T} X_t$
2. Chaining method: extension to covering numbers.

Dudley's inequality is not tight up to a log factor.
A tight bound?

Intuition (Genetic chaining)

Chaining: we choose fixed ϵ_k for every covering set.
Genetic chaining: use adaptive ϵ_k

$$\varepsilon_k = \sup_{t \in T} d(t, T_k),$$

Chaining: 用 ϵ_k 找covering set T_k
Genetic chaining: 用covering set T_k 找 ϵ_k

Generic Chaining Bound

Definition 8.5.1 (Talagrand's γ_2 functional)**.** Let (T, d) be a metric space. A sequence of subsets $(T_k)_{k=0}^{\infty}$ of T is called an *admissible sequence* if the cardinalities of T_k satisfy (8.40). The γ_2 *functional* of T is defined as

$$\gamma_2(T, d) = \inf_{(T_k)} \sup_{t \in T} \sum_{k=0}^{\infty} 2^{k/2} d(t, T_k)$$

$$|T_0| = 1, \quad |T_k| \leq 2^{2^k}, \quad k = 1, 2, \ldots \tag{8.40}$$

Theorem 8.5.3 (Generic chaining bound)**.** *Let $(X_t)_{t \in T}$ be a mean zero random process on a metric space (T, d) with sub-gaussian increments as in* (8.1)*. Then*

$$\mathbb{E} \sup_{t \in T} X_t \leq C K \gamma_2(T, d).$$

Hint: the proof is similar.

Example

Exercise 8.5.2 **(γ_2 functional and Dudley's sum).** ☕☕☕ Consider the same set $T \subset \mathbb{R}^n$ as in Exercise 8.1.12, i.e.

$$T := \left\{ \frac{e_k}{\sqrt{1 + \log k}}, \; k = 1, \dots, n \right\}.$$

(a) Show that the γ_2-functional of T (with respect to the Euclidean metric) is bounded, i.e.

$$\gamma_2(T, d) = \inf_{(T_k)} \sup_{t \in T} \sum_{k=0}^{\infty} 2^{k/2} d(t, T_k) \leq C.$$

Hint: Use the first 2^{2^k} vectors in T to define T_k.

(b) Check that Dudley's sum is unbounded, i.e.

$$\inf_{(T_k)} \sum_{k=0}^{\infty} 2^{k/2} \sup_{t \in T} d(t, T_k) \to \infty$$

Tight!

Theorem 8.6.1 (Talagrand's majorizing measure theorem). *Let $(X_t)_{t \in T}$ be a mean zero Gaussian process on a set T. Consider the canonical metric defined on T by (7.13), i.e. $d(t,s) = \|X_t - X_s\|_{L^2}$. Then*

$$c \cdot \gamma_2(T,d) \le \mathbb{E} \sup_{t \in T} X_t \le C \cdot \gamma_2(T,d).$$

Other extension (c.f., Sudakov-Fernique's inequality in Gaussian Process)

Corollary 8.6.2 (Talagrand's comparison inequality). *Let $(X_t)_{t \in T}$ be a mean zero random process on a set T and let $(Y_t)_{t \in T}$ be a mean zero Gaussian process. Assume that for all $t, s \in T$, we have*

$$\|X_t - X_s\|_{\psi_2} \leq K \|Y_t - Y_s\|_{L^2}.$$

Then

$$\mathbb{E} \sup_{t \in T} X_t \leq CK \, \mathbb{E} \sup_{t \in T} Y_t.$$

Application: Chevnet's inequality

Theorem 8.7.1 (Sub-gaussian Chevet's inequality)**.** *Let A be an $m \times n$ random matrix whose entries A_{ij} are independent, mean zero, sub-gaussian random variables. Let $T \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^m$ be arbitrary bounded sets. Then*

$$\mathbb{E} \sup_{x \in T,\, y \in S} \langle Ax, y \rangle \le CK \left[w(T)\, \mathrm{rad}(S) + w(S)\, \mathrm{rad}(T) \right]$$

where $K = \max_{ij} \|A_{ij}\|_{\psi_2}$.

$$\mathbb{E} \sup_{x \in T,\, y \in S} \langle Ax, y \rangle \ge c \left[w(T)\, \mathrm{rad}(S) + w(S)\, \mathrm{rad}(T) \right].$$

Take-away Messages

1. Generic Chaining Bound with γ_2
   - Tight bound for $\mathbb{E} \sup_{t \in T} X_t$
   - Harder to calculate than Dudley's inequality
   - Extension to Sudakov-Fernique's inequality

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Thanks!