HDP
(24) Escaping Theorem

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前情回顾

1. Matrix Deviation Inequality: random process, concentration, geometry

- General

$$\mathbb{E}\sup_{x\in T}\left|\|Ax\|_2-\sqrt{m}\|x\|_2\right|\leq CK^2\gamma(T).$$

Setting $T=S^{n-1}$: random matrix concentration
Setting $T=T-T$: random projection
Setting $T=\Sigma^{1/2}S^{n-1}$ (square version): covariance estimation with stable rank
Setting $T=\mathcal{X}$: JL lemma on infinite set

M* bound

Theorem 9.4.2 (M^* bound). *Consider a set $T \subset \mathbb{R}^n$. Let A be an $m \times n$ matrix whose rows A_i are independent, isotropic and sub-gaussian random vectors in $\mathbb{R}^n$. Then the random subspace $E = \ker A$ satisfies*

$$\mathbb{E}\,\mathrm{diam}(T \cap E) \leq \frac{CK^2 w(T)}{\sqrt{m}},$$

where $K = \max_i \|A_i\|_{\psi_2}$.

- Proof Hint: set x and y in matrix deviation inequality to ker(A) [A(x-y)=0]
- Note: M* bound is related to random projection.

Escaping Theorem

Theorem 9.4.7 (Escape theorem). *Consider a set $T \subset S^{n-1}$. Let A be an $m \times n$ matrix whose rows A_i are independent, isotropic and sub-gaussian random vectors in $\mathbb{R}^n$. If*

$$\boxed{m \geq CK^4 w(T)^2,} \tag{9.17}$$

then the random subspace $E = \ker A$ satisfies

$$T \cap E = \emptyset$$

with probability at least $1 - 2\exp(-cm/K^4)$. Here $K = \max_i \|A_i\|_{\psi_2}$.

- Proof Hint: Use high-prob version of matrix deviation inequality and choose a proper probability.
- A similar results can be proved in finite set rotation (c.f., random subspace).

Take-away Messages

M* bound: For the intersect of a set T and a random space, its diam can be very small w.h.p.

Escaping Theorem: with a large m, the intersection can be empty w.h.p.

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Thanks!