高维概率

High-Dimensional Probability

六、次高斯随机向量

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- **上节课说了啥**

1. 高维空间比起低维有一些非平凡性质：维度灾难
2. Isotropy的定义：非伸缩旋转
3. Isotropy的性质：几乎正交
4. Isotropy的好处：引入随机性

- **这节课要说啥**

1. 什么是次高斯随机向量
2. 一些随机向量的例子

- 次高斯随机向量：用低维定义高维

Definition 3.4.1 (Sub-gaussian random vectors). A random vector X in $\mathbb{R}^n$ is called *sub-gaussian* if the one-dimensional marginals $\langle X, x\rangle$ are sub-gaussian random variables for all $x \in \mathbb{R}^n$. The *sub-gaussian norm* of X is defined as

$$\|X\|_{\psi_2} = \sup_{x \in S^{n-1}} \| \langle X, x\rangle \|_{\psi_2}.$$

- 一个小性质：从坐标角度出发

Lemma 3.4.2 (Sub-gaussian distributions with independent coordinates). *Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent, mean zero, sub-gaussian coordinates X_i. Then X is a sub-gaussian random vector, and*

$$\|X\|_{\psi_2} \le C \max_{i \le n} \|X_i\|_{\psi_2}.$$

- 一些小例子——Multivariate Gaussian distribution

$X \sim N(\mu, \Sigma)$

$$f_X(x) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} e^{-(x-\mu)^\mathsf{T}\Sigma^{-1}(x-\mu)/2}, \quad x \in \mathbb{R}^n$$

➢ **次高斯性**

Exercise 3.3.4 (Characterizat ribution). ☕☕☕ Let X be a random vector in $\mathbb{R}^n$. Show that $\|X\|_{\psi_2} \leq C.$ iate normal distribution if and only if every one-dimensional marginal $\langle X, \theta \rangle$, $\theta \in \mathbb{R}^n$, has a (univariate) normal distribution.

- 一些小例子——Multivariate Gaussian distribution

$X \sim N(\mu, \Sigma)$

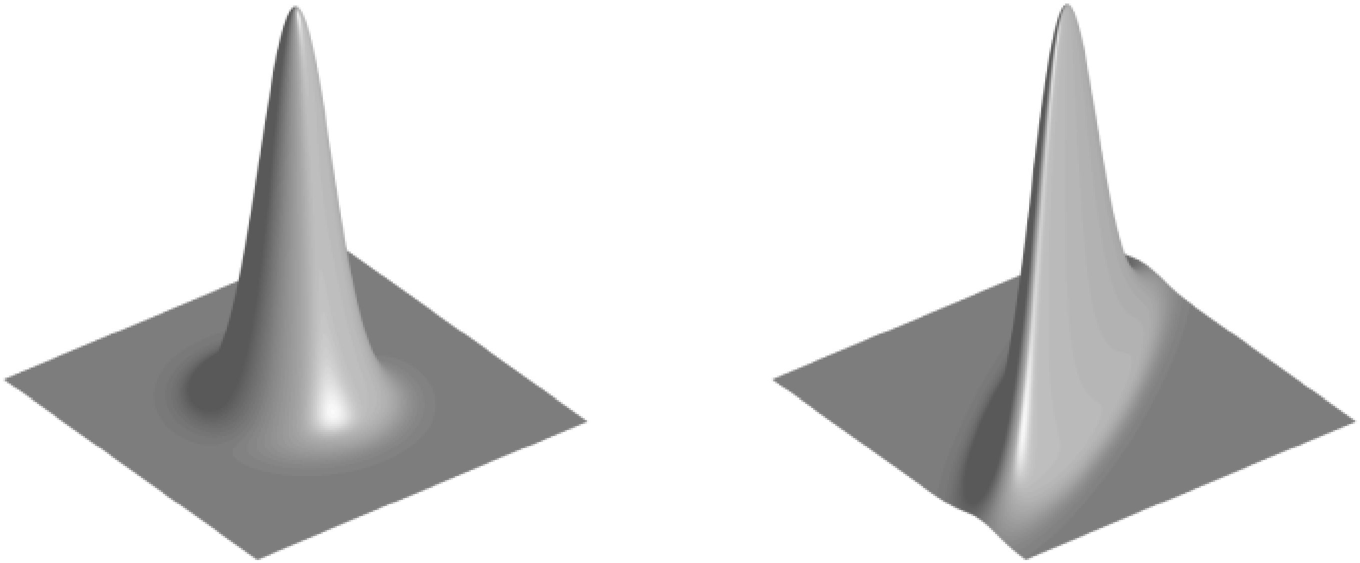

Figure 3.5 The densities of the isotropic distribution $N(0, I_2)$ and a non-isotropic distribution $N(0, \Sigma)$.

- 一些小例子——Symmetric Bernoulli Distribution

$$X \sim \mathrm{Unif}\left(\{-1,1\}^n\right)$$

- Discrete
- Coordinates are independent
- Isotropic

➢ 次高斯模

$$\|X\|_{\psi_2} \leq C.$$

- 一些小例子——Spherical Distribution

$$X \sim \mathrm{Unif}\left(\sqrt{n}\, S^{n-1}\right)$$

- Continuous
- Coordinates are not independent
- Isotropic

➢ 次高斯模

$$\|X\|_{\psi_2} \leq C.$$

- 一些小例子——Gaussian and Spherical

$$N(0, I_n) \approx \text{Unif}\left(\sqrt{n}S^{n-1}\right)$$

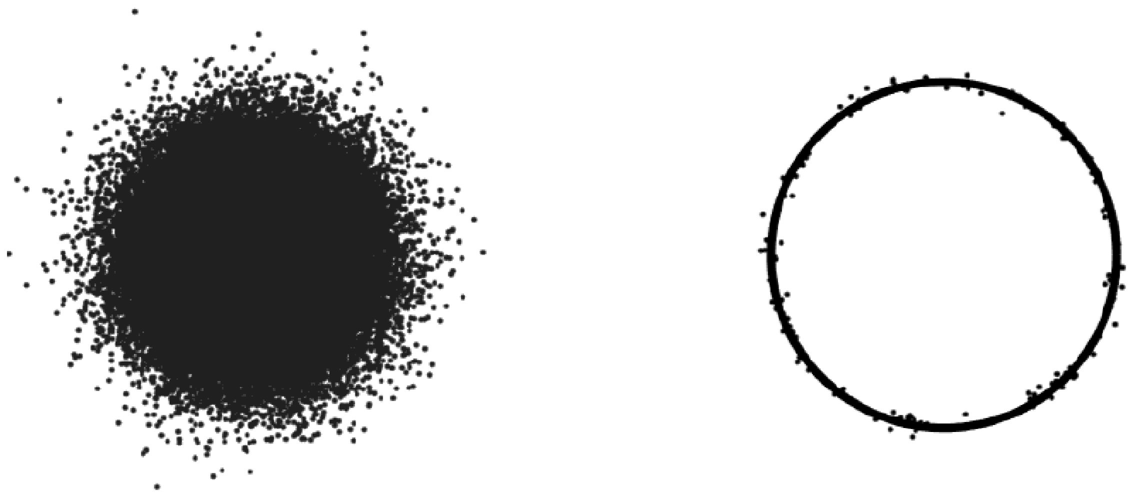

Figure 3.6 A Gaussian point cloud in two dimensions (left) and its intuitive visualization in high dimensions (right). In high dimensions, the standard normal distribution is very close to the uniform distribution on the sphere of radius $\sqrt{n}$.

- 一些小例子——coordinate distribution (Frames)

$$X \sim \mathrm{Unif}\{\sqrt{n}\, e_i : i = 1, \ldots, n\}$$

- Discrete
- Coordinates are independent
- Isotropic

Of all high-dimensional distributions, Gaussian is often the most convenient to prove results for, so we may think of it as "the best" distribution. The coordinate distribution, the most discrete of all distributions, is "the worst".

➢ 次高斯模

$$\|X\|_{\psi_2} \asymp \sqrt{\frac{n}{\log n}}.$$

- 一些小例子——Isotropic Convex

$$X \sim \text{Unif}(K)$$

$$Z \sim \text{Unif}(\Sigma^{-1/2}K)$$

- Continuous
- Isotropic
- Not always sub-Gaussian

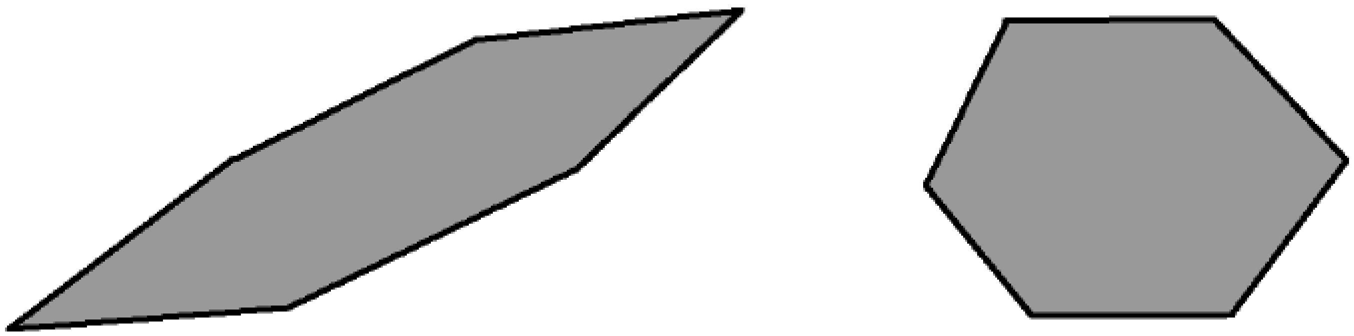

Figure 3.8 A convex body K on the left is transformed into an isotropic convex body TK on the right. The pre-conditioner T is computed from the covariance matrix Σ of K as $T = \Sigma^{-1/2}$.

1. 什么是次高斯随机向量？

和任意普通向量的内积都是次高斯

2. 一些随机向量的例子

Gaussian, Spherical, Symmetric Bernoulli,
Coordinate, Isotropic Convex

Isotropic? Sub-Gaussian? Independent coordinate?

谢谢！

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