高维概率

High-Dimensional Probability

八、应用：格罗滕迪克不等式

@滕佳烨

- 上节课说了啥

1. Coordinate-wise sub-Gaussian向量有Concentration结果

$$\|X\| \approx \sqrt{n}$$

$$\|BX\| \approx \|B\|_F$$

2. 普通的次高斯向量只有下界没上界

- **这节课要说啥**

1. 格罗滕迪克不等式
2. 整数规划的半正定规划近似

- **格罗滕迪克不等式**

Theorem 3.5.1 (Grothendieck's inequality). *Consider an $m \times n$ matrix (a_{ij}) of real numbers. Assume that, for any numbers $x_i, y_j \in \{-1, 1\}$, we have*

$$\left| \sum_{i,j} a_{ij} x_i y_j \right| \le 1.$$

Then, for any Hilbert space H and any vectors $u_i, v_j \in H$ satisfying $\|u_i\| = \|v_j\| = 1$, we have

$$\left| \sum_{i,j} a_{ij} \langle u_i, v_j \rangle \right| \le K,$$

where $K \le 1.783$ is an absolute constant.

- 注意到，上面是离散的{-1, 1}，下面<u, v>的取值[-1, 1]
- 这不是一个平凡结果，不能直接取$(x_i, y_i) = sign(< u_i, v_j >)$
- 怎么跟概率没关系呢？---证明过程里

- **格罗滕迪克不等式**

Theorem 3.5.1 (Grothendieck's inequality). *Consider an $m \times n$ matrix (a_{ij}) of real numbers. Assume that, for any numbers $x_i, y_j \in \{-1, 1\}$, we have*

$$\Big|\sum_{i,j} a_{ij} x_i y_j\Big| \le 1.$$

Then, for any Hilbert space H and any vectors $u_i, v_j \in H$ satisfying $\|u_i\| = \|v_j\| = 1$, we have

$$\Big|\sum_{i,j} a_{ij} \langle u_i, v_j \rangle\Big| \le K,$$

where $K \le 1.783$ is an absolute constant.

- **这不是一个平凡结果, 为什么不能直接取$(\mathbf{x_i}, \mathbf{y_j}) = \mathbf{sign}(<\mathbf{u_i}, \mathbf{v_j}>)$**？注意到只有$n+m$个$\mathrm{u_j}, \mathrm{v_j}$, 却有nm项相乘，因此他们之间是相关的！有可能你让其中一个(x_{i_1}, y_{j_1})满足，另一个地方(x_{i_1}, y_{j_2})就不满足了！

- **格罗滕迪克不等式**

Theorem 3.5.1 (Grothendieck's inequality). *Consider an $m \times n$ matrix (a_{ij}) of real numbers. Assume that, for any numbers $x_i, y_j \in \{-1, 1\}$, we have*

$$\left|\sum_{i,j} a_{ij} x_i y_j\right| \le 1.$$

Then, for any Hilbert space H and any vectors $u_i, v_j \in H$ satisfying $\|u_i\| = \|v_j\| = 1$, we have

$$\left|\sum_{i,j} a_{ij} \langle u_i, v_j \rangle\right| \le K,$$

where $K \le 1.783$ is an absolute constant.

- Hint

1. 首先引入随机性，$< u_i, v_j >=\mathrm{E}\, U_i V_j, U =< g, u_i >$ 。
2. 考虑到U_i, V_j大概率都很小，将其进行截断，小的部分直接bound（配合条件），大的部分概率又很低，定理即得证。
3. 这种引入随机性后截断的技巧还是比较有用的。这样得到的K=288

还可以用核技巧证明$K \le 1.783$

- 半正定规划

Definition 3.5.4. A *semidefinite program* is an optimization problem of the following type:

$$\text{maximize } \langle A, X\rangle : \quad X \succeq 0, \quad \langle B_i, X\rangle = b_i \text{ for } i = 1, \ldots, m. \tag{3.18}$$

Here A and B_i are given $n \times n$ matrices and b_i are given real numbers. The running "variable" X is an $n \times n$ positive-semidefinite matrix, indicated by the notation $X \succeq 0$. The inner product

$$\langle A, X\rangle = \operatorname{tr}(A^\mathsf{T} X) = \sum_{i,j=1}^{n} A_{ij} X_{ij} \tag{3.19}$$

is the canonical inner product on the space of $n \times n$ matrices.

- 这是凸优化问题！

- 考虑一个整数规划问题

$$\text{maximize} \sum_{i,j=1}^{n} A_{ij} x_i x_j : \quad x_i = \pm 1 \text{ for } i = 1, \ldots, n$$

- 这是NP-hard问题！ -> 给出其近似(approximation)形式

$$\text{maximize} \sum_{i,j=1}^{n} A_{ij} \langle X_i, X_j \rangle : \quad \|X_i\|_2 = 1 \text{ for } i = 1, \ldots, n.$$

- 近似的效果怎么样?

Theorem 3.5.6. *Let* INT(A) *denote the maximum in the integer optimization problem* (3.20) *and* SDP(A) *denote the maximum in the semidefinite problem* (3.21). *Then*

$$\mathrm{INT}(A) \le \mathrm{SDP}(A) \le 2K \cdot \mathrm{INT}(A)$$

where $K \le 1.783$ *is the constant in Grothendieck's inequality.*

1. **格罗滕迪克不等式**

连接离散和连续的桥梁（之一）

2. **整数规划的半正定近似**

使用半正定规划近似整数规划，其近似效果能够得到保障！

$$\mathrm{INT}(A) \leq \mathrm{SDP}(A) \leq 2K \cdot \mathrm{INT}(A)$$

谢谢！

@滕佳烨